國立臺灣科技大學
九十二學年度博士班招生考試試題
系所組別：工業管理系博士班丁組
科　目：　機率與統計

總分100分

(25 %)

1. Let f(x, y) = 6cx be the joint density function of two random variables X and Y with 0 < x < y < c.
   (5 %) (1) Find the appropriate value of c.
   (5 %) (2) Find the marginal distribution of X.
   (7 %) (3) Find the conditional probability $\Pr(Y < \frac{1}{3} \mid X < \frac{1}{2})$.
   (8 %) (4) Find the density function of Z = X − Y.

(10 %)

2. Let X be a (uniformly) discrete random variable with probability mass function $\Pr(X = x) = \frac{1}{b-a}$ for $x = a+1, a+2, \ldots, b$, where a and b are integer values with $a < b$. Prove that $\mathrm{Var}(X) = \frac{(b-a)^2 - 1}{12}$ remains the same for all possible situations ($a < 0, b < 0$), ($a < 0, b > 0$), and ($a > 0, b > 0$).

(15 %)

3. If there is a random sample of size n selected from the uniform distribution with density function $f(x) = \frac{1}{\theta}$ for $0 < x < \theta$ and $\theta > 0$.
   (5 %) (a) What is the maximum likelihood estimator $\hat{\theta}$ of θ?
   (5 %) (b) Derive the density function of $\hat{\theta}$.
   (5 %) (c) Find the value of c such that $E(c\hat{\theta}) = \theta$.

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4. 15% (PROOF) **(Chebyshev's Inequality)** *If the random variable X has a mean μ and variance σ^2, then for every $k \geq 1$,*

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Definition Let $f_{Y|x}(y)$ denote the pdf of the random variable Y for a given value x, and let $E(Y|x)$ denote the expected value associated with $f_{Y|x}(y)$. The function

$$y = E(Y|x)$$

is called the *regression curve of Y on x*.

Definition introduces the notion of a regression curve in the most general of contexts. In practice, there is one special case of the function $y = E(Y|x)$ that is particularly important. Known as the *simple linear model*, it makes four assumptions:

(1) $f_{Y|x}(y)$ is a normal pdf for all x.
(2) The standard deviation, σ, associated with $f_{Y|x}(y)$ is the same for all x.
(3) The means of all the conditional Y-distributions are collinear—that is,

$$y = E(Y|x) = \beta_0 + \beta_1 x$$

(4) All of the conditional distributions represent independent random variables.

5. 15% (PROOF) Let $(x_1, Y_1), (x_2, Y_2), \ldots,$ and (x_n, Y_n) be a set of points satisfying the simple linear model, $E(Y|x) = \beta_0 + \beta_1 x$. The maximum-likelihood estimators for β_0, β_1, and σ^2 are given by

$$\hat{\beta}_1 = \frac{n\sum_{i=1}^{n} x_i Y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

where $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i, i = 1, \ldots, n$.

6. 20% (PROOF) Let $(x_1, Y_1), (x_2, Y_2), \ldots,$ and (x_n, Y_n) be a set of points satisfying the simple linear model, $E(Y|x) = \beta_0 + \beta_1 x$. Let $\hat{\beta}_0, \hat{\beta}_1$, and $\hat{\sigma}^2$ be the MLEs for β_0, β_1, and σ^2, respectively. Then

(a) $\hat{\beta}_0$ and $\hat{\beta}_1$ are both normally distributed.
(b) $\hat{\beta}_0$ and $\hat{\beta}_1$ are both unbiased: $E(\hat{\beta}_0) = \beta_0$ and $E(\hat{\beta}_1) = \beta_1$.
(c) $\operatorname{Var}(\hat{\beta}_1) = \dfrac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$
(d) $\operatorname{Var}(\hat{\beta}_0) = \dfrac{\sigma^2 \sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sigma^2\left[\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$