國立臺灣科技大學

九十三學年度博士班考試試題

系所組別：工業管理系丁組

科　　目：機率與統計

*Show intermediate steps and formulas for partial credit. You must explain how you compute your results or answers for full credit.*

總分 100 分

1. (20 points)

Let $f(x,y)=e^{-y}$ for $0<x<y<\infty$ be the joint p.d.f. of the two continuous random variables $X$ and $Y$, and $M(t_1,t_2)=E(e^{t_1X+t_2Y})$ be the moment-generating function of the joint distribution if it exists.

(a) (10 points) Using $M(t_1,t_2)$, derive the moment-generating functions of the marginal distributions of $X$ and $Y$, respectively.

(b) (10 points) Using moment-generating functions, compute $E(X)$, $E(Y)$, and $Cov(X,Y)$.

2. (15 points)

Suppose that $E(Y|X)=aX+b$ for some constants $a$ and $b$. The mean and variance of the random variable $X$ are $\mu$ and $\sigma^2$, respectively. Find the value of $E(XY)$ in terms of $a$, $b$, $\mu$, and $\sigma^2$.

3. (15 points)

Suppose that $X_1$ and $X_2$ are independent random variables each having the probability distribution

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & x>0\\ 0 & elsewhere.\end{cases}$$

Let $Y_1=X_1+X_2$ and $Y_2=\dfrac{X_1}{X_1+X_2}$. What is the joint probability distribution of $Y_1$ and $Y_2$?

4. (25 points)

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu,\sigma^2)$ with both parameters unknown. Use $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$, $S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$, and the values $t_{(u,\alpha)}$ and $\chi^2_{(v,\alpha)}$ to answer the following questions, where $t_{(u,\alpha)}$ and $\chi^2_{(v,\alpha)}$ are the $t$-value and $\chi^2$-value satisfying $\Pr(T \ge t_{(u,\alpha)})=\alpha$ and $\Pr(\chi^2 \ge \chi^2_{(v,\alpha)})=\alpha$ with degrees of freedom $u$ and $v$, respectively.

(a) (5 points) Determine the 95% equal-tail confidence interval for $\mu$.

(b) (5 points) Determine the 95% equal-tail confidence interval for $\sigma^2$.

(c) (5 points) Compute the expected interval length of $\sigma^2$ in (b).

(d) (10 points) Utilize the Chebyshev's inequality with $n=500\sigma^2$ to show that $\Pr(\overline{X}-\mu \ge 0.1)\le 0.1$.

5. (25 points)

Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution with density function

$$f(x)=\frac{1}{2\theta},\ -\theta\le x\le\theta,\ \theta>0.$$

(a) (5 points) Find the maximum likelihood estimator $\hat{\theta}$ of $\theta$.

(b) (10 points) Show your process to get the probability density function of $\hat{\theta}$.

(c) (10 points) Find the value of $c$ such that $E(c\hat{\theta})=\theta$.